Bijective and general arithmetic codings for Pisot automorphisms of the torus*

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Abstract

Let T be an algebraic automorphism of $\mathbb{T}^m$ having the following property: the characteristic polynomial of its matrix is irreducible over $\mathbb{Q}$, and a Pisot number β is one of its roots. We define the mapping $\varphi_{\mathbf{t}}$ acting from the two-sided β-compactum onto $\mathbb{T}^m$ as follows:

$$\varphi_{\mathbf{t}}(\bar{\varepsilon}) = \sum_{k\in\mathbb{Z}} \varepsilon_k T^{-k}\mathbf{t},$$

where $\mathbf{t}$ is a fundamental homoclinic point for T, i.e., a point homoclinic to $\mathbf{0}$ such that the linear span of its orbit is the whole homoclinic group (provided such a point exists). We call such a mapping an arithmetic coding of T. This paper is aimed to show that under some natural hypothesis on β (which is apparently satisfied for all Pisot units) the mapping $\varphi_{\mathbf{t}}$ is bijective a.e. with respect to the Haar measure on the torus. Besides, we study the case of more general parameters $\mathbf{t}$, not necessarily fundamental, and relate the number of preimages of $\varphi_{\mathbf{t}}$ to certain number-theoretic quantities. We also give several full criteria for T to admit a bijective arithmetic coding. This work continues the study begun in [24] for the special case $m = 2$.

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1 Introduction

Let T be an algebraic automorphism of the torus $\mathbb{T}^m$ given by a matrix $M \in GL(m, \mathbb{Z})$ with the following property: the characteristic polynomial for M is irreducible over $\mathbb{Q}$, and a Pisot number $\beta > 1$ is its root (we recall that an algebraic integer is called *a Pisot number* if it is greater than 1 and all its Galois conjugates are less than 1 in modulus). Since $\det M = \pm 1$, β is a *unit*, i.e., an invertible element of the ring $\mathbb{Z}[\beta] = \mathbb{Z}[\beta^{-1}]$. We will call such an automorphism a *Pisot automorphism*. Note that since none of the eigenvalues of M lies on the unit circle, T is hyperbolic.

Our goal is to present a symbolic coding of T which, roughly speaking, reveals not just the structure of T itself but the natural action of the torus on itself as well. Let us give more precise definitions.

Let X_β denote the two-sided *β-compactum*, i.e., the space of all *admissible* two-sided sequences in the alphabet $\{0, 1, \ldots, [\beta]\}$. More precisely, a representation of an $x \in [0, 1)$ of the form

$$x = \pi(\varepsilon_1, \varepsilon_2, \ldots) := \sum_1^\infty \varepsilon_k \beta^{-k} \tag{1}$$

is called the *β-expansion* of x if the "digits" $\{\varepsilon_k\}_1^\infty$ are obtained by means of the greedy algorithm (similarly to the decimal expanions), i.e., $\varepsilon_1 = \varepsilon_1(x) = [\beta x]$, $\varepsilon_k = \varepsilon_k(x) = [\beta \tau^k(x)]$, where $\tau(x) = \{\beta x\} := \beta x \bmod 1$. The set of all possible sequences $\{\{\varepsilon_k(x)\}_1^\infty : x \in [0, 1)\}$ is called the (one-sided) *β-compactum* and denoted by X_β^+. A sequence whose tail is 0^∞ will be called *finite*.

The β-compactum can be described more explicitly. Let $1 = \sum_1^\infty d'_k \beta^{-k}$ be the expansion of 1 defined as follows: $d'_1 = [\beta]$, $d'_n = [\beta \tau^n 1]$, $n \geq 2$. If the sequence $\{d'_n\}$ is not finite, we put $d_n \equiv d'_n$. Otherwise let $k = \max\{j : d'_j > 0\}$, and $(d_1, d_2, \ldots) := (\overline{d'_1, \ldots, d'_{k-1}, d'_k - 1})$, where the bar denotes the period of a purely periodic sequence.

We will write $\{x_n\}_1^\infty \prec \{y_n\}_1^\infty$ if $\{x_n\}_1^\infty \neq \{y_n\}_1^\infty$ and $x_n < y_n$ for the smallest $n \geq 1$ such that $x_n \neq y_n$. Then by definition,

$$X_\beta^+ = \{\{\varepsilon_n\}_1^\infty : (\varepsilon_n, \varepsilon_{n+1}, \ldots) \prec (d_1, d_2, \ldots) \text{ for all } n \in \mathbb{N}\}$$

(see [17]). Similarly, we define the *two-sided β-compactum* as

$$X_\beta = \{\{\varepsilon_n\}_{-\infty}^\infty : (\varepsilon_n, \varepsilon_{n+1}, \ldots) \prec (d_1, d_2, \ldots) \text{ for all } n \in \mathbb{Z}\}.$$

Both compacta are naturally endowed with the weak topology, i.e. with the topology of coordinate-wise convergence, as well as with the natural shifts. Let the *β-shift*

$\sigma_\beta : X_\beta \to X_\beta$ act as follows: $\sigma_\beta(\bar\varepsilon)_k = \varepsilon_{k+1}$, and σ_β^+ be the corresponding one-sided shift on X_β^+. For a Pisot β the properties of the β-shift are well-studied. Its main property is that it is *sofic*, i.e., is a projection of a subshift of finite type. In fact this is equivalent to $\{d_n\}_1^\infty$ being eventually periodic (see, e.g., the review [6]).

We extend the β-expansions to the nonnegative integers in the usual way (similarly to the decimal expansions).

Lemma 1 *(see [4], [20]) Any nonnegative element of the ring $\mathbb{Z}[\beta]$ has an eventually periodic β-expansion if β is a Pisot number.*

There is a natural operation of addition in X_β, namely, if both sequences $\bar\varepsilon$ and $\bar\varepsilon'$ are *finite to the left* (i.e., there exists $N \in \mathbb{Z}$ such that $\varepsilon_k = \varepsilon'_k = 0,\ k \le N$), then by definition, $\bar\varepsilon' + \bar\varepsilon = \bar\varepsilon''$ such that $\sum_k \varepsilon''_k \beta^{-k} = \sum_k (\varepsilon'_k + \varepsilon_k)\beta^{-k}$. Later we will show that under some natural assumption on β this operation can be extended to sequences which are not necessarily finite to the left.

Let $Fin(\beta)$ denote the set of nonnegative x's whose β-expansions are finite. Obviously, $Fin(\beta) \subset \mathbb{Z}[\beta]_+ := \mathbb{Z}[\beta] \cap \mathbb{R}_+$, but the inverse inclusion does not hold for an arbitrary Pisot unit.

Definition 2 *A Pisot unit β is called **finitary** if*

$$Fin(\beta) = \mathbb{Z}[\beta]_+.$$

A large class of Pisot numbers considered in [11] is known to have this property. A practical algorithm for checking whether a given Pisot number is finitary was suggested in [1]. Here is a simple example showing that not every Pisot unit is finitary. Let $r \ge 3$, and $\beta^2 = r\beta - 1$. Then $X_\beta = \{\bar\varepsilon : 0 \le \varepsilon_k \le r-1,\ (\varepsilon_k, \dots, \varepsilon_{k+n}) \ne (r-1, r-2, \dots, r-2, r-1),\ k \in \mathbb{Z}, n \ge 1\}$ and $1 - \beta^{-1} = (r-2)\beta^{-1} + (r-2)\beta^{-2} + \dots$, i.e., β is not finitary.

Definition 3 *A Pisot unit β is called **weakly finitary** if for any $\delta > 0$ and any $x \in \mathbb{Z}[\beta]_+$ there exists $f \in Fin(\beta) \cap [0, \delta)$ such that $x + f \in Fin(\beta)$ as well.*

This condition was considered in the recent work by Sh. Akiyama [3], in which the author shows that the boundary of the natural sofic tiling generated by a weakly finitary Pisot β has Lebesgue measure zero (moreover, these conditions are actually equivalent). The author is grateful to Sh. Akiyama for drawing his attention to this paper and for helpful discussions.

A slightly weaker (but possibly equivalent) condition

$$\mathbb{Z}[\beta] = Fin(\beta) - Fin(\beta)$$

together with the finiteness of $\{d'_n\}$ was used in the recent Ph.D. dissertation [13] to show that the spectrum of the Pisot substitutional dynamical system

$$0 \to 10^{d'_1}, 1 \to 20^{d'_2}, \dots, l-2 \to (l-1)0^{d'_{l-1}}, l-1 \to 0^{d'_l},$$

(where $l = \max\{n : d'_n \neq 0\}$), is purely discrete. This claim is a generalization of the corresponding result for a finitary β from [26] (see also [27]).

Conjecture 4 *Any Pisot unit is weakly finitary.*

To support this conjecture, we are going to explain how to verify that a particular Pisot unit is weakly finitary. Firstly, one needs to describe all the elements of the set

$$Z_\beta = \{\alpha \in \mathbb{Z}[\beta] \cap [0,1) : \alpha \text{ has a purely periodic } \beta\text{-expansion}\}. \tag{2}$$

Lemma 5 *(see [3]). The set Z_β is finite.*

Proof. The sketch of the proof is as follows: basically, the claim follows from Lemma 10, which implies that the denominator of any $\alpha \in \mathcal{P}_\beta$ in the standard basis of $\mathbb{Q}(\beta)$ is uniformly bounded, whence the period of the β-expansion of α is bounded as well. ■

Therefore, we have a finite collection of numbers $\{\sum_{j=0}^{m-1} y_j\beta^j : |y_j| \le q\}$ to "check for periods" (here q is the denominator of ξ_0 defined by (6) in the standard basis of the ring). Next, it is easy to see that if suffices to check that Definition 3 holds for any $x = \alpha \in Z_\beta$ (see [3]). Moreover, we can confine ourselves to the case $f \in Fin(\beta) \cap [\beta^{-2p}, \beta^{-p})$, where p is the period of α. Indeed, if such an f exists, $\beta^{-p}f$ will do as well, and we will be able to make f arbitrarily small. All known examples of Pisot units prove to be weakly finitary.

We will need the following technical result.

Lemma 6 *A Pisot unit is weakly finitary if and only if the following condition is satisfied: there exists $\eta = \eta(\beta) \in (0,1)$ such that for any $\delta > 0$ and any $x \in \mathbb{Z}[\beta]_+$ there exists $f \in Fin(\beta) \cap [\eta\delta, \delta)$ such that $x + f \in Fin(\beta)$ as well.*

Proof. It suffices to show that if β is weakly finitary, then η in question does exist. Let β be weakly finitary; then for any $\alpha \in Z_\beta$ there exists $f_\alpha \in Fin(\beta)$ such that $\alpha + f_\alpha \in Fin(\beta)$. Let α has the β-expansion $(\overline{\alpha_1, \dots, \alpha_p})$ and $\alpha^* = \sum_1^p \alpha_j\beta^{-j}$. Without loss of generality we may regard p to be greater than the preperiod + the period of the sequence $\{d_n\}_1^\infty$ (as p is not necessarily the *smallest* period of α). Since f_α can be made arbitrarily small, we may fix it such that

$$\alpha + f_\alpha < \alpha^* + \beta^{-p}\alpha. \tag{3}$$

Put $\eta := \min\{f_\alpha : \alpha \in Z_\beta\}$.

Let $x \in \mathbb{Z}[\beta]_+$. By Lemma 1 the β-expansion of x is eventually periodic, and splitting it into the preperiodic and periodic parts, we have $x = x_0 + \beta^{-k}\alpha,\ x_0 \in Fin(\beta), k \in \mathbb{Z}, \alpha \in Z_\beta$. Let for simplicity of notation $k = 0$ (the whole picture is shift-invariant). It will suffice to check the condition for $\delta = \delta_n = \beta^{-pn}$. Put $f = f_n := \beta^{-pn} f_\alpha$. Then

$$x + f = (x_0 + \alpha^* + \beta^{-p}\alpha^* + \cdots + \beta^{-(n-1)p}\alpha^*) + \beta^{-pn}(\alpha + f_\alpha) \tag{4}$$

The expression in brackets in (4) belongs to $Fin(\beta)$ and so does the second term. In view of (3) and the definition of X_β the whole sum in (4) belongs to $Fin(\beta)$ as well, because by our choice of p we have necessarily $(\alpha_1, \dots, \alpha_p) \prec (d_1, \dots, d_p)$. Since Z_β is finite and the construction depends on α only, we are done. ■

2 Formulation of the main result and first steps of the proof

We recall that the hyperbolicity of T implies that it has the stable and unstable foliation and consequently the set of homoclinic points. More precisely, a point $\mathbf{t} \in \mathbb{T}^m$ is called *homoclinic to zero* or simply *homoclinic* if $T^n\mathbf{t} \to \mathbf{0}$ as $n \to \pm\infty$ (as is well known, the convergence to $\mathbf{0}$ in this case will be at exponential rate). In other terms, a homoclinic point $\mathbf{t}$ must belong to the intersection of the leaves of the stable foliation L_s and the unstable foliation L_u passing through $\mathbf{0}$. Let $\mathcal{H}(T)$ denote the set of all homoclinic points for T; obviously, $\mathcal{H}(T)$ is a group under addition. In [29] it was shown that every homoclinic point can be obtained by applying the following procedure: take a point $\mathbf{n} \in \mathbb{Z}^m$ and project it onto L_u along L_s. Let $\mathbf{s}$ denote this projection; finally, project $\mathbf{s}$ onto the torus by taking the fractional parts of all its coordinates. The correspondence $\mathbf{n} \leftrightarrow \mathbf{s} \leftrightarrow \mathbf{t}$ is one-to-one. We will call $\mathbf{s} = \mathbf{s}(\mathbf{t})$ the $\mathbb{R}^m$-*coordinate* of a homoclinic point $\mathbf{t}$ and $\mathbf{n}$ the $\mathbb{Z}^m$-*coordinate* of $\mathbf{t}$. Note that since T is a Pisot automorphism, we have $\dim L_u = 1,\ \dim L_s = m - 1$.

We wish to find an *arithmetic coding* φ of T in the following sense: we choose X_β as a symbolic compact space and impose the following restrictions on a map $\varphi : X_\beta \to \mathbb{T}^m$:

1. φ is continuous and bounded-to-one;
2. $\varphi\sigma_\beta = T\varphi$;
3. $\varphi(\bar\varepsilon + \bar\varepsilon') = \varphi(\bar\varepsilon) + \varphi(\bar\varepsilon')$ for any pair of sequences finite to the left.

In [24] it was shown that if $m = 2$, then there exists $\mathbf{t} \in \mathcal{H}(T)$ such that $\varphi = \varphi_{\mathbf{t}} : X_\beta \to \mathbb{T}^m$:

$$\varphi_{\mathbf{t}}(\bar{\varepsilon}) = \sum_{k \in \mathbf{Z}} \varepsilon_k T^{-k} \mathbf{t}. \tag{5}$$

The proof for an arbitrary m is basically the same, and we will omit it. Our primary goal is to find an arithmetic coding that is bijective a.e. Let us make some remarks.

Note that the idea of using homoclinic points to "encode" ergodic toral automorphisms had been suggested by A. Vershik in [28] for $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and was later developed for a more general context in numerous works – see [29], [15], [23], [24], [21]. The choice of X_β as a "coding space" is special in the case in question; indeed, the topological entropy of the shift σ_β is known to be $\log \beta$ and so is the entropy of T. In a more general context (for example, if M has two eigenvalues outside the unit disc) it is still unclear, which compactum might replace X_β. Indeed, since φ is bounded-to-one, the topological entropy of the subshift on this compactum must have the same topological entropy as T, i.e., $\log \prod_{|\beta_j|>1} |\beta_j|$, where $\beta_j,\ j = 1, \dots, m$, are the conjugates of β, and there is apparently no natural subshift associated with β which has this entropy. However, it is worth noting that the *existence* of such compacta in different settings has been shown in [29], [14], [21].

Return to our context. The mapping $\varphi_{\mathbf{t}}$ defined by (5) is indeed well defined and continuous, as the series (5) converges at exponential rate. Furthermore, since $T^k \mathbf{t} = \beta \mathbf{t} \bmod \mathbb{Z}^m$, we have by continuity $\varphi_{\mathbf{t}} \sigma_\beta = T \varphi_{\mathbf{t}}$, i.e., $\varphi_{\mathbf{t}}$ does semiconjugate the shift and a given automorphism T.

We will call $\varphi_{\mathbf{t}}$ a *general arithmetic coding* for T (parametrised by a homoclinic point $\mathbf{t}$).

Lemma 7 *For any choice of* $\mathbf{t}$ *the mapping* $\varphi_{\mathbf{t}}$ *is bounded-to-one.*

Proof. Let $\|\cdot\|$ denote the distance to the closest integer, $\mathbf{s}$ be the $\mathbb{R}^m$-coordinate of $\mathbf{t}$ and $\tilde{T}$ denote the linear transformation of $\mathbb{R}^m$ defined by the matrix M. Let $\varphi_{N,\mathbf{t}}$ be the mapping acting from X_β into $\mathbb{R}^m$ by the formula

$$\phi_{N,\mathbf{t}}(\bar{\varepsilon}) := \sum_{-N}^{N} \varepsilon_k (\tilde{T})^{-k} \mathbf{s}.$$

Then by (5),

$$\varphi_{\mathbf{t}}(\bar{\varepsilon}) = \lim_{N \to +\infty} (\phi_{N,\mathbf{t}}(\bar{\varepsilon}) \bmod \mathbb{Z}^m),$$

where $(x_1, \dots, x_m) \bmod \mathbb{Z}^m = (\{x_1\}, \dots, \{x_m\})$. Therefore, it suffices to show that the diameters of the sets $\phi_{N,\mathbf{t}}(X_\beta)$ are uniformly bounded for all N. We have (recall that $0 \le \varepsilon_k \le [\beta]$):

$$\max\{\|\phi_{N,\mathbf{t}}(\overline{\varepsilon})\| : \overline{\varepsilon} \in X_\beta\} \le [\beta] \left\| \sum_{-N}^{N} (\tilde{T})^{-k}\mathbf{s} \right\| \le [\beta] \sum_{-N}^{N} \left\| (\tilde{T})^{-k}\mathbf{s} \right\|$$
$$\le \text{const} \cdot \sum_{0}^{N} \theta^k < \infty,$$

where $\theta \in (0,1)$ is the maximum of the absolute values of the conjugates of β that do not coincide with β. This proves the lemma. ■

Let the characteristic equation for β be

$$\beta^m = k_1\beta^{m-1} + k_2\beta^{m-2} + \cdots + k_m$$

and T_β denote the toral automorphism given by the *companion matrix* M_β for β, i.e.,

$$M_\beta = \begin{pmatrix} k_1 & k_2 & \dots & k_{m-1} & k_m \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

We first assume the following conditions to be satisfied:

1. T is algebraically conjugate to T_β, i.e., there exists a matrix $C \in GL(m, \mathbb{Z})$ such that $CM = M_\beta C$ (notation: $T \sim T_\beta$).

2. A homoclinic point $\mathbf{t}$ is *fundamental*, i.e., $\langle T^n\mathbf{t} \mid n \in \mathbb{Z}\rangle = \mathcal{H}(T)$.

3. β is weakly finitary.

The notion of fundamental homoclinic point for general actions of expansive group automorphisms was introduced in [16] (see also [21]).

Remark 8 *Note that the second condition implies the first, as the mere existence of a fundamental homoclinic point means that $T \sim T_\beta$ (see Theorem 28 below). Conversely, if $T \sim T_\beta$, then there is always a fundamental homoclinic point for T. Indeed, let $\mathbf{n}_0 = (0, 0, \dots, 0, 1)$ be the $\mathbb{Z}^m$-coordinate of $\mathbf{t}_0$. Then $\mathbf{t}_0$ is a fundamental for T_β and if $CM = M_\beta C$, then $C^{-1}\mathbf{t}_0$ is fundamental for T.*

Now we are ready to formulate the main theorem of the present paper.

Theorem 9 *Provided the above conditions are satisfied, the mapping $\varphi_{\mathbf{t}}$ defined by (5) is bijective a.e. with respect to the Haar measure on the torus.*

Remark. In [24] the claim of the theorem was shown for $m = 2$. We wish to follow the line of exposition of that paper, though it is worth stressing that our approach will be completely different (rather arithmetic than geometric). In [21] this claim was proven for any finitary β and it was conjectured that it holds for any Pisot automorphism satisfying conditions 1 and 2 above. We give further support for this conjecture, as Theorem 9 implies that we actually reduced it to a general number-theoretic conjecture verifiable for any given Pisot unit β (see Conjecture 4).

The rest of the section as well as the next section will be devoted to the proof of Theorem 9; in the last section we will discuss the case when conditions 1 and 2 are not necessarily satisfied.

We are going to need the following number-theoretic claim. Let

$$\mathcal{P}_\beta := \{\xi : \|\xi\beta^n\| \to 0,\ n \to +\infty\}.$$

It is obvious that $\mathcal{P}_\beta$ is a group under addition.

Lemma 10 *There exists $\xi_0 \in \mathbb{Q}(\beta) \setminus \mathbb{Z}[\beta]$ such that*

$$\mathcal{P}_\beta = \xi_0 \cdot \mathbb{Z}[\beta]. \tag{6}$$

Proof. By the well-known result, for any Pisot β, $\xi \in \mathcal{P}_\beta \Leftrightarrow \xi \in \mathbb{Q}(\beta)$ and $Tr(\beta^k\xi) \in \mathbb{Z},\ k \geq k_0$ (where $Tr(\varsigma)$ denotes the trace of an element ς of the extension $\mathbb{Q}(\beta)$, i.e., the sum of all its Galois conjugates) – see, e.g., [8]. Since β is a unit, $Tr(\varsigma) \in \mathbb{Z}$ implies $Tr(\beta^{-1}\varsigma) \in \mathbb{Z}$, whence

$$\mathcal{P}_\beta := \{\xi \in \mathbb{Q}(\beta) : Tr(a\xi) \in \mathbb{Z}\ \forall a \in \mathbb{Z}[\beta]\}. \tag{7}$$

Thus, if we regard $\mathbb{Z}[\beta]$ as a lattice over $\mathbb{Z}$, then by (7), $\mathcal{P}_\beta$ is by definition the dual lattice for $\mathbb{Z}[\beta]$. Hence by the well known ramification theorem (see, e.g., [10, Chapter III]) the equality (6) follows with $\xi_0 = 1/g'(\beta)$, where $g(x) = x^m - k_1x^{m-1} - \cdots - k_m$. ■

We will divide the proof of the main theorem into several steps.

Step 1 (description of the homoclinic group).

Lemma 11 *Any homoclinic point $\mathbf{t}$ for T_β has the $\mathbb{R}^m$-coordinate*

$$\mathbf{s}(\mathbf{t}) = \xi_0 u(1, \beta^{-1}, \dots, \beta^{-m+1}), \tag{8}$$

where $u \in \mathbb{Z}[\beta]$.

Proof. We have $M_\beta \bar v_\beta = \beta \bar v_\beta$, where $\bar v_\beta = (1, \beta^{-1}, \dots, \beta^{-m+1})$. As was mentioned above, the dimension of the unstable foliation L_u is 1, whence $\mathbf{s}(\mathbf{t}) = k\bar v_\beta$, and since $T_\beta^n \mathbf{t} \to \mathbf{0}$, we have $\|k\beta^n\| \to 0$, i.e., $k \in \mathcal{P}_\beta$. Now the claim of the lemma follows from (6). ■

Let $\mathcal{U}_\beta$ denote the group of units (= invertible elements) of the ring $\mathbb{Z}[\beta]$.

Lemma 12 *There is a one-to-one correspondence between the group $\mathcal{U}_\beta$ and the set of fundamental homoclinic points for T_β. Namely, if $\mathbf{t}$ is fundamental, then u in (8) is a unit and vice versa.*

Proof. Suppose $\mathbf{t}$ is fundamental. Then the homoclinic point $\mathbf{t}_0$ whose $\mathbb{R}^m$-coordinate is $\mathbf{s}_0 = (\xi_0, \xi_0\beta^{-1}, \dots, \xi_0\beta^{-m+1})$ can be represented as a finite linear integral combination of the powers $T^k\mathbf{t}$, i.e.,

$$\xi_0(1, \beta^{-1}, \dots, \beta^{-m+1}) = \sum_k e_k \beta^k \xi_0 u(1, \beta^{-1}, \dots, \beta^{-m+1}),$$

whence $u\sum_k e_k\beta^k = 1$. Therefore, u is invertible in the ring $\mathbb{Z}[\beta]$.

Conversely, if $u \in \mathcal{U}_\beta$, then using the same method, we show that the claim of the lemma follows from the fact that the equation $ux = u'$ always has the solution in $\mathbb{Z}[\beta]$, namely, $x = u^{-1}u'$. ■

Step 2 (reduction to $T = T_\beta$). To prove Theorem 9, we may without loss of generality assume $T = T_\beta$. Indeed, suppose $M = C^{-1}M_\beta C$, where $C \in GL(m, \mathbb{Z})$. Then there is a natural one-to-one correspondence between $\mathcal{H}(T)$ and $\mathcal{H}(T_\beta)$, namely, $\mathbf{t} \in \mathcal{H}(T) \Leftrightarrow C\mathbf{t} \in \mathcal{H}(T_\beta)$. Furthermore, if $\varphi_\mathbf{t}$ is bijective a.e., then so is $\varphi_{C\mathbf{t}}$, as $\varphi_{C\mathbf{t}} = C\varphi_\mathbf{t}$.

So, we assume first that $T = T_\beta$, and $\mathbf{t}$ is a general fundamental homoclinic point for T_β given by (8). In this case the formula (5) becomes

$$\varphi_\mathbf{t}(\bar\varepsilon) = \lim_{N\to+\infty} \sum_{k=-N}^{+\infty} \varepsilon_k \beta^{-k} \begin{pmatrix} \xi_0 u \\ \xi_0 u\beta^{-1} \\ \vdots \\ \xi_0 u \beta^{-m+1} \end{pmatrix} \bmod \mathbb{Z}^m.$$

Step 3 (the preimage of $\mathbf{0}$). Let Z_β be defined by (2).

Lemma 13 *The preimage of $\mathbf{0}$ can be described as follows:*

$$\mathcal{O}_\beta := \varphi_\mathbf{t}^{-1}(\mathbf{0}) = \{\bar\varepsilon \in X_\beta : \bar\varepsilon \textit{ is purely periodic and } \sum_1^\infty \varepsilon_j\beta^{-j} \in Z_\beta\}.$$

Proof. By Lemma 7, $\mathcal{O}_\beta$ is finite and since it is shift-invariant, it must contain purely periodic sequences only. Let $\alpha = \sum_1^\infty \varepsilon_j \beta^{-j}$. Then by (2), $\|\alpha u \xi_0 \beta^n\| \to 0$ as $n \to \infty$, whence from (6), therefore, $\alpha u \in \mathbb{Z}[\beta]$, and $\alpha \in \mathbb{Z}[\beta]$, because $u \in \mathcal{U}_\beta$. ■

Step 4 (description of the full preimage of any point of the torus). We are going to show that $\varphi_{\mathbf{t}}$ is "linear" in the sense that for any two sequences $\bar{\varepsilon}, \bar{\varepsilon}' \in \varphi_{\mathbf{t}}^{-1}(x)$ their "difference" will belong to $\mathcal{O}_\beta$. More precisely, let $\varepsilon^{(N)}$ denote both the sequence $(\dots, 0, 0, \dots, 0, \varepsilon_{-N}, \varepsilon_{-N+1}, \dots)$ and its "value" $e^{(N)} := \sum_{k=-N}^{+\infty} \varepsilon_k \beta^{-k}$

Lemma 14 *If $\varphi_{\mathbf{t}}(\bar{\varepsilon}) = \varphi_{\mathbf{t}}(\bar{\varepsilon}')$, then for any $N \geq 1$ there exists $\alpha \in Z_\beta$ such that*

$$|e^{(N)} - (e')^{(N)}| = \beta^N \alpha.$$

Proof. Let $\mathcal{E}$ denote the set of all partial limits of the collection of sequences $|\varepsilon^{(N)} - (\varepsilon')^{(N)}|, N \geq 1$, where $|\varepsilon^{(N)} - (\varepsilon')^{(N)}|$ is the sequence $(\dots, 0, 0, \dots, 0, \varepsilon''_{-N}, \varepsilon''_{-N+1}, \dots)$ whose "value" is $|e^{(N)} - (e')^{(N)}|$. It suffices to show that $\mathcal{E} \subset \mathcal{O}_\beta$. Let $\bar{\delta} \in \mathcal{E}$; by definition, there exists a sequence of positive integers $\{N_k\}$ such that $\delta^{(N_k)} = \left|(\varepsilon')^{(N_k)} - \varepsilon^{(N_k)}\right|$, $k = 1, 2, \dots$ Then $\varphi_{\mathbf{t}}(\bar{\delta}) = \lim_{k\to\infty} \varphi_{\mathbf{t}}(\delta^{(N_k)}) = \mathbf{0}$, and we are done. ■

Therefore, if $\bar{\varepsilon} \in \varphi_{\mathbf{t}}^{-1}(x)$ for some $x \in \mathbb{T}^m$, then we know that to obtain any $\bar{\varepsilon}' \in \varphi_{\mathbf{t}}^{-1}(x)$, one may take one of the partial limits of the sequence $\{\varepsilon^{(N)} + \beta^N \alpha\}$ for $\alpha \in Z_\beta$, perhaps, depending on N. We will write

$$\bar{\varepsilon} \sim \bar{\varepsilon}' \text{ iff } \varphi_{\mathbf{t}}(\bar{\varepsilon}) = \varphi_{\mathbf{t}}(\bar{\varepsilon}'). \tag{9}$$

Conclusion. Thus, we reduced the proof of Theorem 9 to a certain claim about the two-sided β-compactum.

Basically, our goal now is to show that the procedure described above will not change an arbitrarily long tail of a generic sequence $\bar{\varepsilon} \in X_\beta$ and therefore, will not change $\bar{\varepsilon}$ itself.

3 Final steps of the proof and examples

Let μ_β denote the measure of maximal entropy for the shift (X_β, σ_β), and μ_β^+ be its one-sided analog. We wish to prove that

$$\mu_\beta\{\bar{\varepsilon} \in X_\beta : \#[\bar{\varepsilon}] = 1\} = 1, \tag{10}$$

where $[\bar{\varepsilon}] = \{\bar{\varepsilon}' \in X_\beta : \bar{\varepsilon}' \sim \bar{\varepsilon}\}$.

Step 5 (estimation of the measure of the "bad" set). We will need some basic facts about the measure μ_β. For technical reasons we prefer to deal with its one-sided analog μ_β^+.

Lemma 15 *There exists a constant $C_1 = C_1(\beta) \in (0,1)$ such that for any $n \geq 2$ and any $(i_1, i_2, \dots) \in X_\beta^+$,*

$$\mu_\beta^+ (\varepsilon_n = i_n \mid \varepsilon_{n-1} = i_{n-1}, \dots, \varepsilon_1 = i_1) \geq C_1.$$

Proof. Let the mapping $\pi : X_\beta^+ \to [0,1)$ be given by formula (1) and $m_\beta^+ = \pi(\mu_\beta^+)$. Let $C_n(\bar\varepsilon) = (\varepsilon_n = i_n, \varepsilon_{n-1} = i_{n-1}, \dots, \varepsilon_1 = i_1) \subset X_\beta^+$ and $\Delta_n(\bar\varepsilon) = \pi(C_n(\bar\varepsilon))$. The Garsia Separation Lemma [12] says that there exists a constant $K = K(\beta) > 0$ such that if $\bar\varepsilon$ and $\bar\varepsilon'$ are two sequences in X_β^+ and $\sum_{k=1}^n \varepsilon_k \beta^{-k} \neq \sum_{k=1}^n \varepsilon'_k \beta^{-k}$, then $\left|\sum_{k=1}^n (\varepsilon_k - \varepsilon'_k)\beta^{-k}\right| \geq K\beta^{-n}$. Hence

$$K \leq \beta^n \mathcal{L}_1(\Delta_n(\bar\varepsilon)) \leq 1,$$

where $\mathcal{L}_1$ denotes the Lebesgue measure on $[0,1]$. Since for any $\beta > 1$, m_β^+ is equivalent to $\mathcal{L}_1$ and the corresponding density is uniformly bounded away from 0 and ∞ (see [18]), we have for some $K' > 1$,

$$1/K' \leq \beta^n m_\beta^+(\Delta_n(\bar\varepsilon)) \leq K',$$

whence by the fact that π is one-to-one except for a countable set of points,

$$1/K' \leq \beta^n \mu_\beta^+(C_n(\bar\varepsilon)) \leq K'$$

and the claim of the lemma holds with $C_1 = (\beta K')^{-2}$. ■

There is a natural arithmetic structure on X_β^+: the sum of two sequences $\bar\varepsilon$ and $\bar\varepsilon'$ is defined as the sequence equal to the β-expansion of the sum $\{\sum_1^\infty (\varepsilon_k + \varepsilon'_k)\beta^{-k}\}$. Let $X_\beta^{(n)}$ denote the set of finite words of length n that are extendable to a sequence in X_β^+ by writing noughts at all places starting with $n+1$. We will sometimes identify $X_\beta^{(n)}$ with the set $Fin_n(\beta) := \{\bar\varepsilon : \varepsilon_k \equiv 0,\ k \geq n+1\}$.

By the sum $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) + \bar\varepsilon'$, we will imply $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, 0, 0, \dots) + \bar\varepsilon'$. In [11] it was shown that there exists a natural $L_1 = L_1(\beta)$ such that if $\bar\varepsilon \in Fin_n(\beta)$, $\bar\varepsilon' \in Fin_n(\beta)$ and $\bar\varepsilon + \bar\varepsilon' \in Fin(\beta)$, then $\bar\varepsilon + \bar\varepsilon' \in Fin_{n+L_1}(\beta)$.

Recall that by Lemma 6 there exists $\eta = \eta(\beta) \in (0,1)$ such that the quantity f in Definition 3 can be chosen in $(\eta\delta, \delta)$ instead of $(0, \delta)$. We set

$$L_2 := \frac{\log(1/\eta)}{\log\beta}.$$

Let $L := \max\{L_1, L_2\}$. We can reformulate the hypothesis that β is weakly finitary as follows ($\bar\alpha$ denotes the β-expansion of α):

$$\begin{gathered}\text{for any } (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in X_\beta^{(n)} \text{ there exists } (\varepsilon_{n+1}, \dots, \varepsilon_{n+L}) \in X_\beta^{(L)} : \\ (\varepsilon_1, \dots, \varepsilon_{n+L}) \in X_\beta^{(n+L)},\ \bar\alpha + (\varepsilon_1, \dots, \varepsilon_{n+L}) \in Fin(\beta) \text{ for all } \alpha \in Z_\beta.\end{gathered} \tag{11}$$

A direct consequence of Lemma 15 is

Corollary 16 *For any* $(i_1, i_2, \dots) \in X_\beta^+$,

$$\frac{\mu_\beta^+(\varepsilon_{n+L} = i_{n+L}, \varepsilon_{n+L-1} = i_{n+L-1}, \dots, \varepsilon_1 = i_1)}{\mu_\beta^+(\varepsilon_n = i_n, \varepsilon_{n-1} = i_{n-1}, \dots, \varepsilon_1 = i_1)} \ge C_2 = C_1^L. \tag{12}$$

Lemma 17 *If* $(i_1, \dots, i_n) \in X_\beta^{(n)}$ *and* $(j_1, \dots, j_k) \in X_\beta^{(k)}$*, then* $(i_1, \dots, i_n, 0, 0, 0, 0, j_1, \dots, j_k) \in X_\beta^{(n+k+4)}$.

Proof. The claim follows from the definition of X_β (see Introduction) and the fact that the positive root β_0 of the equation $x^3 = x + 1$ is the smallest Pisot number [9]. Indeed, $\beta_0^5 = \beta_0^4 + 1$ and X_{β_0} is a subshift of finite type, namely,

$$X_{\beta_0} = \left\{ \bar\varepsilon \in \prod_{-\infty}^{+\infty} \{0,1\} \mid \varepsilon_n = 1 \Rightarrow \varepsilon_{n+1} = \varepsilon_{n+2} = \varepsilon_{n+3} = \varepsilon_{n+4} = 0 \right\}.$$

Now the desired claim follows from [17, Lemma 3] asserting that if $\beta' < \beta$, then $(d_1(\beta'), d_2(\beta'), \dots) \prec (d_1(\beta), d_2(\beta), \dots)$. ∎

Let

$$\begin{aligned}
\mathfrak{A} &= \{\bar\varepsilon \in X_\beta^+ \mid \exists n \in \mathbb{N} : \forall \alpha \in Z_\beta,\ \bar\alpha + (\varepsilon_1, \dots, \varepsilon_n) \in Fin(\beta)\},\\
\mathfrak{A}_n &= \{\bar\varepsilon \in X_\beta^+ \mid \forall \alpha \in Z_\beta,\ \bar\alpha + (\varepsilon_1, \dots, \varepsilon_n) \in Fin(\beta)\},\\
\mathfrak{A}' &= \{\bar\varepsilon \in X_\beta^+ \mid \exists n \in \mathbb{N} : \varepsilon_{n+1} = \dots = \varepsilon_{n+L+4} = 0\}.
\end{aligned}$$

We will write $tail(\bar\varepsilon) = tail(\bar\varepsilon')$ if there exists $n \in \mathbb{N}$ such that $\varepsilon_k = \varepsilon'_k,\ k \ge n$. The meaning of the above definitions consists in the fact that if $\bar\varepsilon \in \mathfrak{A} \cap \mathfrak{A}'$, then $\bar\varepsilon \in \mathfrak{A}_n \cap \mathfrak{A}'$ for some $n \ge 1$ and by the theorem from [11] mentioned above, $(\varepsilon_1, \dots, \varepsilon_n) + \bar\alpha = (\varepsilon'_1, \dots, \varepsilon'_{n+L})$, whence by Lemma 17

$$tail(\bar\varepsilon + \bar\alpha) = tail(\bar\varepsilon)$$

(more precisely, the tail will stay unchanged starting with the $(n+L+1)$'th symbol). It is obvious that $\mathfrak{A} = \cup_n \mathfrak{A}_n$. We wish to prove that $\mu_\beta^+(\mathfrak{A} \cap \mathfrak{A}') = 1$. By the ergodicity of $(X_\beta^+, \mu_\beta^+, \sigma_\beta^+)$, we have $\mu_\beta^+(\mathfrak{A}') = 1$, it suffices to show that $\mu_\beta^+(\mathfrak{A}) = 1$. Let $\mathfrak{B}_n = X_\beta^+ \setminus \mathfrak{A}_n$.

Proposition 18 *There exists a constant* $\gamma = \gamma(\beta) \in (0,1)$ *such that*

$$\mu_\beta^+\left(\bigcap_{k=1}^n \mathfrak{B}_k\right) \le \gamma^n. \tag{13}$$

Proof. We have

$$\mu_\beta^+(\mathfrak{B}_1 \cap \mathfrak{B}_2 \cap \ldots \cap \mathfrak{B}_n) = \mu_\beta^+(\mathfrak{B}_1) \cdot \prod_{k=2}^{n} \mu_\beta^+(\mathfrak{B}_k \mid \mathfrak{B}_{k-1} \cap \ldots \cap \mathfrak{B}_1)$$
$$\leq \prod_{k=2}^{n} \mu_\beta^+(\mathfrak{B}_k \mid \mathfrak{B}_{k-1} \cap \ldots \cap \mathfrak{B}_1).$$

Since

$$\prod_{j=k-L}^{k} \mu_\beta^+(\mathfrak{B}_j \mid \mathfrak{B}_{j-1} \cap \ldots \cap \mathfrak{B}_1) = \frac{\mu_\beta^+(\mathfrak{B}_k \cap \ldots \cap \mathfrak{B}_1)}{\mu_\beta^+(\mathfrak{B}_{k-L-1} \cap \ldots \cap \mathfrak{B}_1)}$$
$$\leq \frac{\mu_\beta^+(\mathfrak{B}_k \cap \mathfrak{B}_{k-L-1} \cap \mathfrak{B}_{k-L-2} \cap \ldots \cap \mathfrak{B}_1)}{\mu_\beta^+(\mathfrak{B}_{k-L-1} \cap \ldots \cap \mathfrak{B}_1)}$$
$$= \mu_\beta^+(\mathfrak{B}_k \mid \mathfrak{B}_{k-L-1} \cap \mathfrak{B}_{k-L-2} \cap \ldots \cap \mathfrak{B}_1),$$

we have

$$\mu_\beta^+(\mathfrak{B}_1 \cap \mathfrak{B}_2 \cap \ldots \cap \mathfrak{B}_n) \leq \prod_{k=2}^{[n/L]} \mu_\beta^+(\mathfrak{B}_{Lk} \mid \mathfrak{B}_{Lk-L-1} \cap \mathfrak{B}_{Lk-L-2} \cap \ldots \cap \mathfrak{B}_1). \tag{14}$$

Now by the formula (12), β being weakly finitary (see (11)) and the deifnition of L we have

$$\mu_\beta^+(\mathfrak{A}_{k+L} \mid \varepsilon_k = i_k, \ldots, \varepsilon_1 = i_1) \geq C_2 > 0$$

for any $(i_1, \ldots, i_k) \in X_\beta^{(k)}$. Hence

$$\mu_\beta^+(\mathfrak{B}_{Lk} \mid \mathfrak{B}_{Lk-L-1} \cap \mathfrak{B}_{Lk-L-2} \cap \ldots \cap \mathfrak{B}_1) \leq 1 - C_2,$$

and from (14) we finally obtain the estimate

$$\mu_\beta^+(\mathfrak{B}_1 \cap \mathfrak{B}_2 \cap \ldots \cap \mathfrak{B}_n) \leq (1 - C_2)^{n/L},$$

whence one can take $\gamma = (1 - C_2)^{1/L}$, and (13) is proven. ∎

As a consequence we obtain the following claim about the irrational rotations of the circle by the elements of $\mathbb{Z}[\beta]$. Let, as above, $\bar{\alpha}$ denote the β-expansion of α.

Theorem 19 *For a weakly finitary Pisot unit β and any $\alpha \in \mathbb{Z}[\beta] \cap [0, 1)$ we have*

$$tail(\bar{\varepsilon} + \bar{\alpha}) = tail(\bar{\varepsilon})$$

for μ_β^+-a.e. $\bar{\varepsilon} \in X_\beta^+$.

Proof. We showed that $\mu_\beta^+(\cap_1^\infty \mathfrak{B}_n) = 0$, whence $\mu_\beta^+(\mathfrak{A}) = \mu_\beta^+(\cup_1^\infty \mathfrak{A}_n) = 1$. ■

Conclusion of the proof of Theorem 9. Fix $k \in \mathbb{N}$. To complete the proof of Theorem 9, it suffices to show that the set

$$\mathcal{D}^{(k)} = \{\bar\varepsilon \in X_\beta \mid \varepsilon_j \equiv \varepsilon_j', \ j \geq k \ \forall \bar\varepsilon' \sim \bar\varepsilon\},$$

has the full measure μ_β. By Proposition 18, for

$$\mathcal{D}_N^{(k)} = \{(\varepsilon_{-N}, \varepsilon_{-N+1}, \dots) \in X_\beta^+ \mid (\dots 0, 0, \varepsilon_{-N}, \varepsilon_{-N+1}, \dots) \in \mathcal{D}^{(k)}\},$$

$\mu_\beta^+(\mathcal{D}_N^{(k)}) \geq 1 - \gamma^{k-N} \to 1$ as $N \to +\infty$. Hence

$$\mu_\beta(\mathcal{D}^{(k)}) = \lim_{N \to +\infty} \mu_\beta^+(\mathcal{D}_N^{(k)}) = 1,$$

and therefore

$$\mu_\beta\left(\bigcap_{k=1}^{\infty} \mathcal{D}^{(k)}\right) = 1$$

which implies (10). We have thus shown that for μ_β-a.e. $\bar\varepsilon \in X_\beta$, $\#\varphi_{\mathbf{t}}^{-1}(\varphi_{\mathbf{t}}(\bar\varepsilon)) = 1$. Let $\mathcal{L}$ denote the image of μ_β under $\varphi_{\mathbf{t}}$. Since μ_β is ergodic, so is $\mathcal{L}$ and since $h_{\mu_\beta}(\sigma_\beta) = \log\beta$, we have $h_{\mathcal{L}}(T) = \log\beta$ as well. Hence $\mathcal{L} = \mathcal{L}_m$ is the Haar measure on the torus, as it is the unique ergodic measure of maximal entropy. So, we proved that

$$\mathcal{L}_m\{x \in \mathbb{T}^m \mid \#\varphi_{\mathbf{t}}^{-1}(x) = 1\} = 1,$$

which is the claim of Theorem 9.

As a corollary we obtain the following claim about the arithmetic structure of X_β itself.

Proposition 20 *Let $\sim$ denote the equivalence relation on X_β defined by (9) and $X_\beta' := X_\beta/\sim$. Then X_β' is a group isomorphic to $\mathbb{T}^m$.*

Thus, X_β is an *almost group* in the sense that it suffices to "glue" some k-tuples (for $k < \infty$) within the set of measure zero to turn the two-sided β-compactum for a weakly finitary Pisot unit β into a group (which will be isomorphic to the torus of the corresponding dimension). Note that in dimension 2 this factorisation can be described more explicitly – see [24, Section 1].

The following claim is a generalisation of Theorem 4 from [25]. Let $\mathcal{D}(T)$ denotes the *centraliser* for T, i.e.,

$$\mathcal{D}(T) = \{A \in GL(m, \mathbb{Z}) : AT = TA\}.$$

Proposition 21 *For a Pisot automorphism whose matrix is algebraically conjugate to the corresponding companion matrix there is a one-to-one correspondence between the following sets:*

1. *the fundamental homoclinic points for T;*
2. *the bijective arithmetic codings for T;*
3. *the units of the ring $\mathbb{Z}[\beta]$;*
4. *the centraliser for T;*

Proof. We already know that any bijective arithmetic coding is parametrised by a fundamental homoclinic point. Let $\mathbf{t}_0$ be such a point for T; then any other fundamental homoclinic point $\mathbf{t}$ satsifies $\mathbf{s} = u\mathbf{s}_0$, where $\mathbf{s}_0$ and $\mathbf{s}$ are the corresponding $\mathbb{R}^m$-coordinates and $u \in \mathcal{U}_\beta$ – the proof is essentially the same as in Lemma 12. On the other hand, if $\varphi_\mathbf{t}$ is a bijective arithmetic coding for T, then as easy to see, $A := \varphi_\mathbf{t}\varphi_{\mathbf{t}_0}^{-1}$ is a toral automorphism commuting with T (this mapping is well defined almost everywhere on the torus, hence it can be defined everywhere by continuity). Finally, if $u \in \mathcal{U}_\beta$ and $u = \sum_{j=1}^{m-1} u_j\beta^j$, then $A := \sum_{j=1}^{m-1} u_j M^j$ belongs to $GL(m, \mathbb{Z})$ and commutes with M and vice versa. ■

Example 1. (see [23]) Let T be given by the matrix $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Here β is the golden ratio, and. $\xi_0 = \frac{1}{\sqrt{5}} = \frac{-1+2\beta}{5}$, and

$$\mathcal{U}_\beta = \{\pm\beta^n,\ n \in \mathbb{Z}\}.$$

Any bijective arithmetic coding for T thus will be of the form

$$\varphi(\overline{\varepsilon}) = \lim_{N\to+\infty} \sum_{k=-N}^{+\infty} \varepsilon_k\beta^{-k} \begin{pmatrix} \vartheta\beta^n/\sqrt{5} \\ \vartheta\beta^{n-1}/\sqrt{5} \end{pmatrix} \bmod \mathbb{Z}^2,$$

where $\vartheta \in \{\pm 1\}$ and $n \in \mathbb{Z}$.

For more two-dimensional examples see [24].

Example 2. Let T be given by the matrix $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Here β is the real root of the "tribonacci" equation $x^3 = x^2 + x + 1$; as is well known, β is finitary in this case (see, e.g., [11]). We have $\xi_0 = \frac{1}{-1-2\beta+3\beta^2} = \frac{1+9\beta-4\beta^2}{22}$, and since $\mathbb{Z}[\beta]$ is the maximal order in the field $\mathbb{Q}(\beta)$ and both conjugates of β are complex, again

$$\mathcal{U}_\beta = \{\pm\beta^n,\ n \in \mathbb{Z}\}$$

(recall that by Dirichlet's Theorem, $\mathcal{U}_\beta$ must be "one-dimensional", see, e.g., [7]). Hence any bijective arithmetic coding for T is of the form

$$\varphi(\overline{\varepsilon}) = \lim_{N\to+\infty} \sum_{k=-N}^{+\infty} \varepsilon_k \beta^{-k} \begin{pmatrix} \vartheta \frac{1+9\beta-4\beta^2}{22} \beta^n \\ \vartheta \frac{1+9\beta-4\beta^2}{22} \beta^{n-1} \\ \vartheta \frac{1+9\beta-4\beta^2}{22} \beta^{n-2} \end{pmatrix} \bmod \mathbb{Z}^3,$$

where $\vartheta \in \{\pm 1\}$ and $n \in \mathbb{Z}$.

Example 3. Let $M = \begin{pmatrix} 3 & 4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Here β is the positive root of $x^3 = 3x^2 + 4x + 1$. By the result from [2], β is finitary (see Introduction for the definition) and the fundamental units of the ring are β and $3 + \beta^{-1}$, i.e.,

$$\mathcal{U}_\beta = \{\pm\beta^n, \pm(3+\beta^{-1})^n,\ n \in \mathbb{Z}\}.$$

Besides, $\xi_0 = \frac{1}{3\beta^2-6\beta-4} = \frac{-13-21\beta+6\beta^2}{7}$. Hence any bijective arithmetic coding is either

$$\varphi(\overline{\varepsilon}) = \lim_{N\to+\infty} \sum_{k=-N}^{+\infty} \varepsilon_k \beta^{-k} \begin{pmatrix} \vartheta \frac{-13-21\beta+6\beta^2}{7} \beta^n \\ \vartheta \frac{-13-21\beta+6\beta^2}{7} \beta^{n-1} \\ \vartheta \frac{-13-21\beta+6\beta^2}{7} \beta^{n-2} \end{pmatrix} \bmod \mathbb{Z}^3$$

or

$$\varphi(\overline{\varepsilon}) = \lim_{N\to+\infty} \sum_{k=-N}^{+\infty} \varepsilon_k \beta^{-k} \begin{pmatrix} \vartheta \frac{-13-21\beta+6\beta^2}{7} (3+\beta^{-1})^n \\ \vartheta \frac{-13-21\beta+6\beta^2}{7} (3+\beta^{-1})^{n-1} \\ \vartheta \frac{-13-21\beta+6\beta^2}{7} (3+\beta^{-1})^{n-2} \end{pmatrix} \bmod \mathbb{Z}^3,$$

where $\vartheta \in \{\pm 1\}$ and $n \in \mathbb{Z}$.

Example 4. Finally, let $M = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Here β satisfies $x^4 = x^3 + 1$. Let us show that β is weakly finitary. A direct inspection shows that the only nonzero tail for the positive elements of $\mathbb{Z}[\beta]$ is $\overline{10000}$. Hence $Z_\beta = \{0, \beta^{-2}+\beta^{-3}, \beta^{-3}+\beta^{-4}, \beta^{-4}+\beta^{-5}, \beta^{-5}+\beta^{-6}, \beta^{-6}+\beta^{-7}\}$. Let, for example, $x = \beta^{-2}+\beta^{-3}$; since $x + \beta^{-5} = \beta^{-1}+\beta^{-3} = \beta^{-1}+\beta^{-4}+\beta^{-7} = 1+\beta^{-7} \in Fin(\beta)$, we have by periodicity $x + \beta^{-5n} \in Fin(\beta)$ for any $n \geq 1$. The other cases of $\alpha \in Z_\beta$ are similar. Hence β is weakly finitary and we can apply Theorem 9. It suffices to compute $\mathcal{U}_\beta$; by the Dirichlet Theorem, it must be "two-dimensional" and it is easy to guess that the second fundamental unit (besides β itself) is $1+\beta$. Hence $\mathcal{U}_\beta = \{\pm\beta^n, \pm(1+\beta)^n,\ n \in \mathbb{Z}\}$ and the formula for a bijective arithmetic coding can be derived similarly to the previous examples in view of $\xi_0 = \frac{1}{-3\beta^2+4\beta^3} = \frac{-12-16\beta+73\beta^2+9\beta^3}{283}$.

4 General arithmetic codings and related algebraic results

In this section we will present some results for the case when $\mathbf{t}$ is not necessarily fundamental or T is not algebraically conjugate to the companion matrix automorphism. We will still assume β to be weakly finitary. Let us begin with the case $T = T_\beta$ with a general $\mathbf{t}$. We recall that there is an isomorphism between the homoclinic group $\mathcal{H}(T)$ and the group $\mathcal{P}_\beta$, i.e., $\mathbf{t} \leftrightarrow \xi$. Let $\varphi_\xi : X_\beta \to \mathbb{T}^m$ be defined as usual:

$$\varphi_\xi(\bar{\varepsilon}) = \varphi_{\mathbf{t}}(\bar{\varepsilon}) = \sum_{k \in Z} \varepsilon_k T_\beta^{-k}\mathbf{t} = \lim_{N \to +\infty} \left(\sum_{k=-N}^{\infty} \varepsilon_k \beta^{-k} \right) \begin{pmatrix} \xi \\ \xi\beta^{-1} \\ \vdots \\ \xi\beta^{-m+1} \end{pmatrix} \bmod \mathbb{Z}^m,$$

where $\xi = \xi(\mathbf{t}) \in \mathcal{P}_\beta$. The question is, what will be the value of $\#\varphi_\xi^{-1}(x)$ for a $\mathcal{L}_m$-typical $x \in \mathbb{T}^m$?

The next assertion answers this question; it is a generalization of the corresponding result proven in [24] for $m = 2$ and for a finitary β in [22]. Let $D = D(\beta)$ denote the discriminant of β in the field extension $\mathbb{Q}(\beta)/\mathbb{Q}$, i.e., the product $\prod_{i \neq j}(\beta_i - \beta_j)^2$, where $\{\beta_1 = \beta, \beta_2, \ldots, \beta_m\}$ are the Galois conjugates of β.

Theorem 22 *For an a.e. $x \in \mathbb{T}^m$ with respect to the Haar measure,*

$$\#\varphi_\xi^{-1}(x) \equiv |DN(\xi)|,$$

where $N(\cdot)$ denotes the norm of an element of the extension $\mathbb{Q}(\beta)/\mathbb{Q}$.

Proof. Let φ_0 denote the bijective arithmetic coding for T_β parametrised by ξ_0 and $\ell := \xi/\xi_0 \in \mathbb{Z}[\beta]$. If $\ell = \sum_{i=0}^{m-1} c_i\beta^i$, then one can consider the mapping $A_\xi := \varphi_\xi\varphi_0^{-1} : \mathbb{T}^m \to \mathbb{T}^m$; it will be well defined on the dense set and we may extend it to the whole torus. By the linearity of both maps, A_ξ is a toral endomorphism. Thus, we have

$$\varphi_\xi = A_\xi\varphi_0. \tag{15}$$

Let A'_ξ is given by the formula $A'_\xi = \sum_{i=0}^{m-1} c_i T_\beta^i$. For the basis sequence $f^{(0)} =$

$(\ldots, 0, 0, \ldots, 0, 1, 0, \ldots, 0, 0, \ldots)$ with the unity at the first coordinate we have

$$\begin{aligned}(A_\xi \varphi_0)(f^{(0)}) &= A_\xi(\xi_0, \xi_0\beta^{-1}, \ldots, \xi_0\beta^{-m+1}) \bmod \mathbb{Z}^m \\ &= \sum_{i=0}^{m-1} c_i T^i(\xi_0, \xi_0\beta^{-1}, \ldots, \xi_0\beta^{-m+1}) \bmod \mathbb{Z}^m \\ &= \sum_{i=0}^{m-1} c_i \beta^i(\xi_0, \xi_0\beta^{-1}, \ldots, \xi_0\beta^{-m+1}) \bmod \mathbb{Z}^m \\ &= (\xi, \xi\beta^{-1}, \ldots, \xi\beta^{-m+1}) \bmod \mathbb{Z}^m = \varphi_\xi(f^{(0)}).\end{aligned}$$

Therefore, by the linearity and continuty, we have $A_\xi = A'_\xi = \sum_{i=0}^{m-1} c_i T_\beta^i$. As φ_0 is 1-to-1 a.e., φ_ξ will be K-to-1 a.e. with $K = |\det A_\xi|$. By definition, $N(\ell)$ is the determinant of the matrix of the multiplication operator $x \mapsto \ell x$ in the standard basis of $\mathbb{Q}(\beta)$, whence $N(\ell) = \det A_\xi$, because T_β is given by the companion matrix. Finally, $N(\ell) = N(\xi)/N(\xi_0) = DN(\xi)$, as by the result from [19, Section 2.7], $N(\xi_0) = 1/D$ whenever ξ_0 is as in formula (6). ■

Note that historically the first attempt to find an arithmetic coding for a Pisot automorphism was undertaken in [5]. The author considered the case $T = T_\beta$ and $\mathbf{t}$ given by the $\mathbb{R}^m$-coordinate $\mathbf{s} = (1, \beta^{-1}, \ldots, \beta^{-m+1})$. From the above theorem follows

Corollary 23 *The mapping*

$$\varphi_{\mathbf{t}}(\bar{\varepsilon}) = \lim_{N\to+\infty} \sum_{k=-N}^{+\infty} \varepsilon_k \beta^{-k} \begin{pmatrix} 1 \\ \beta^{-1} \\ \vdots \\ \beta^{-m+1} \end{pmatrix} \bmod \mathbb{Z}^m$$

is $|D|$-to-1 a.e.

Suppose now T is not necessarily algebraically conjugate to T_β. Let M be, as usual, the matrix of T, and for $\mathbf{n} \in \mathbb{Z}^m$ the matrix $B_M(\mathbf{n})$ be defined as follows (we write it column-wise):

$$\begin{aligned}B_M(\mathbf{n}) = (M\mathbf{n}, (M^2 - k_1M)\mathbf{n}, (M^3 - k_1M^2 - k_2M)\mathbf{n}, \ldots, \\ M^{m-1} - k_1M^{m-2} - \cdots - k_{m-2}M)\mathbf{n}, k_m\mathbf{n}).\end{aligned}$$

Lemma 24 *Any integral square matrix satisfying the relation*

$$BM_\beta = MB \tag{16}$$

is $B = B_M(\mathbf{n})$ for some $\mathbf{n} \in \mathbb{Z}^m$.

Proof. Let B be written column-wise as follows: $B = (\mathbf{n}_1, \ldots, \mathbf{n}_m)$. Then by (16) and the definition of M_β,

$$(k_1\mathbf{n}_1 + \mathbf{n}_2, k_2\mathbf{n}_1 + \mathbf{n}_3, \ldots, k_{m-1}\mathbf{n}_1 + \mathbf{n}_m, k_m\mathbf{n}_1) = (M\mathbf{n}_1, \ldots, M\mathbf{n}_m),$$

whence by the fact that $k_m = \pm 1$, we have $B = B_M(\mathbf{n})$ for $\mathbf{n} = k_m\mathbf{n}_m$. ∎

Definition 25 *The integral m-form of m variables defined by the formula*

$$f_M(\mathbf{n}) = \det B_M(\mathbf{n})$$

will be called the ***form associated with*** *T.*

Proposition 26 *Let $\mathbf{t} \in \mathcal{H}(T)$. Then there exists $\mathbf{n} \in \mathbb{Z}^m$ such that*

$$\#\varphi_{\mathbf{t}}^{-1}(x) \equiv |f_M(\mathbf{n})|$$

for $\mathcal{L}_m$-a.e. point $x \in \mathbb{T}^m$. Hence the minimum of the number of preimages for an arithmetic coding of a given automorphism T equals the arithmetic minimum of the associated form f_M.

Proof. Let $\tilde{B} := \varphi_{\mathbf{t}}\varphi_0^{-1}$, where φ_0 is a certain bijective arithmetic coding for T_β. Then $\tilde{B}$ is a linear mapping from $\mathbb{T}^m$ onto itself defined a.e.; let the same letter denote the corresponding toral endomorphism. Then $\tilde{B}T_\beta = \varphi_{\mathbf{t}}\varphi_0^{-1}T_\beta = \varphi_{\mathbf{t}}\sigma_\beta\varphi_0^{-1} = T\varphi_{\mathbf{t}}\varphi_0^{-1} = T\tilde{B}$. Therefore the matrix B of the endomorphism $\tilde{B}$ satisfies (16), whence by Lemma 24, $B = B_M(\mathbf{n})$ for some $\mathbf{n} \in \mathbb{Z}^m$. Hence $\varphi_{\mathbf{t}} = B_M(\mathbf{n})\varphi_0$, and we are done. ∎

Remark 27 *It would be helpful to know whether there is any relationship between the $\mathbf{n}$ in the proposition and the $\mathbb{Z}^m$-coordinate of $\mathbf{t}$.*

Theorem 28 *The following conditions are equivalent:*

1. *An automorphism T admits a bijective arithmetic coding.*
2. *T is algebraically conjugate to T_β.*
3. *The equation*

   $$f_M(\mathbf{n}) = \pm 1$$

   has a solution in $\mathbf{n} \in \mathbb{Z}^m$.

4. There exists a homoclinic point $\mathbf{t}$ *such that for its* $\mathbb{Z}^m$*-coordinate* $\mathbf{n}$,

$$\left\langle M^k \mathbf{n} \mid k \in \mathbb{Z} \right\rangle = \mathbb{Z}^m.$$

Proof. (2)⇒(1): see Remark 8;
(1)⇒(2): see the Proposition 26;
(2)⇔(3): also follows from Proposition 26;
(2)⇔(4): it is obvious that M_β satisfies this property (take $\mathbf{n} = (0, 0, \ldots, 0, 1)$). Hence so does any M which is conjugate to M_β. ∎

Recall that a matrix $M \in GL(m, \mathbb{Z})$ is called *primitive* if there is no matrix $K \in GL(m, \mathbb{Z})$ such that $M = K^n$ for $n \geq 2$. Following [24], we ask the following question: can a Pisot toral automorphism given by a non-primitive matrix admit a bijective arithmetic coding?

Note first that one can simplify the formula for f_M. Namely, since the determinant of a matrix stays unchanged if we multiply one column by some number and add to another column, we have

$$f_M(\mathbf{n}) = \pm \det\,(\mathbf{n}, M\mathbf{n}, \ldots, M^{m-1}\mathbf{n}). \tag{17}$$

Proposition 29 *There exists a sequence of integers* $\mathcal{N}_n(\beta)$ *such that*

$$f_{M^n} = \pm \mathcal{N}_n(\beta) \cdot f_M.$$

More precisely,

$$\mathcal{N}_n(\beta) = \det \begin{pmatrix} a_n^{(1)} & \ldots & a_n^{(m)} \\ a_{2n}^{(1)} & \ldots & a_{2n}^{(m)} \\ \vdots & \ddots & \vdots \\ a_{(m-1)n}^{(1)} & \ldots & a_{(m-1)n}^{(m)} \end{pmatrix},$$

where $\{a_n^{(j)}\}_{j=1}^m$ *are defined as the coefficients of the equation*

$$\beta^n = a_n^{(1)} \beta^{m-1} + a_n^{(2)} \beta^{m-2} + \cdots + a_n^{(m-1)} \beta + a_n^{(m)}.$$

Proof. By (17), the definition of $a_n^{(j)}$ and the Hamilton-Cayley Theorem,

$$\begin{aligned} f_{M^n}(\mathbf{n}) &= \pm \det\,(\mathbf{n}, M^n\mathbf{n}, M^{2n}\mathbf{n}, \ldots, M^{(m-1)n}\mathbf{n}) \\ &= \pm \det \left(\mathbf{n}, \left(\sum_{j=1}^m a_n^{(j)} M^{m-j} \right) \mathbf{n}, \ldots, \left(\sum_{j=1}^m a_{(m-1)n}^{(j)} M^{m-j} \right) \mathbf{n} \right) = \\ &= \pm \mathcal{N}_n(\beta) \cdot \det\,(\mathbf{n}, M\mathbf{n}, M^2\mathbf{n}, \ldots, M^n\mathbf{n}). \end{aligned}$$

∎

Corollary 30 *A non-primitive matrix $M^n \in GL(m, Z)$ is algebraically conjugate to the corresponding companion matrix if and only if so is M, and $\mathcal{N}_n(\beta) = \pm 1$.*

Let us deduce some corollaries for smaller dimensions.

Corollary 31 *(see [24]) For $m = 2$ the automorphism given by a non-primitive matrix M^n admits a bijective arithmetic coding if and only if $n = 2$ and $Tr(M) = \pm 1$.*

Corollary 32 *For $m = 3$ the matrix $K = M^2, M \in GL(3, \mathbb{Z})$, is algebraically conjugate to the corresponding companion matrix if and only if β satisfies one of the following equations:*

1. $\beta^3 = r\beta^2 + 1,\ r \geq 1;$
2. $\beta^3 = r\beta^2 - 1,\ r \geq 3;$
3. $\beta^3 = 2\beta^2 - \beta + 1.$

Proof. We have $\mathcal{N}_2(\beta) = \det \begin{pmatrix} 1 & k_1^2 + k_2 \\ 0 & k_1k_2 + k_3 \end{pmatrix} = k_1k_2 + k_3 = \pm 1$. The case $k_3 = +1$ thus leads to subcases 1 and 3 and $k_3 = -1$ yields subcase 2. ■

Note that if M is the matrix for the "tribonacci automorphism" (see Example 2), then apparently the only power of M that is algebraically conjugate to the corresponding companion matrix, is the cube! Indeed, $\mathcal{N}_2(\beta) = 2, \mathcal{N}_3(\beta) = -1, \mathcal{N}_4(\beta) = -8, \mathcal{N}_5(\beta) = 29$, etc. It seems to be an easy exercise to prove this rigoriously; we leave it to the reader.

Example 5. Let $M = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Here β satisfies $x^3 = 5x^2 - 4x + 1$ and the form associated with M is (we write $\mathbf{n} = (x, y, z)'$)

$$f_M(x, y, z) = x^3 + 2x^2z - xy^2 - xyz + 3xz^2 + y^3 - 3y^2z + 2yz^2 + z^3.$$

Obviously, the Diophantine equation $f_M(x, y, z) = \pm 1$ has a solution, namely, $x = 1, y = z = 0$. Hence by Theorem 28, M is algebraically conjugate to M_β; for example, $B = B_M(1, 0, 0) = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$ conjugates them. To show that T admits a bijective arithmetic coding, it suffices to check that β is weakly finitary. We leave it to the interested reader.

In [24] the author together with A. Vershik considered the case $m = 2$. Here if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then for $\sigma = \det M = \pm 1$,

$$f_M(x, y) = \begin{vmatrix} ax + by & -\sigma x \\ cx + dy & -\sigma y \end{vmatrix} = \sigma(cx^2 - (a - d)xy - by^2),$$

and we related the problem of arithmetic codings to the classical theory of binary quadratic forms. In particular, T admits a bijective arithmetic coding if and only if the Diophantine equation

$$cx^2 - (a - d)xy - by^2 = \pm 1$$

is solvable.

The theory of general m-forms of m variables does not seem to be well developed; nonetheless, we would like to mention a certain algebraic result which looks relevant. Recall that two integral forms are called *equivalent* if there exists a unimodular integral change of variables turning one form into another.

Proposition 33 *Let M_1, M_2 in $GL(m, \mathbb{Z})$ be conjugate, and*

$$AM_1A^{-1} = M_2,$$

where $A \in GL(m, \mathbb{Z})$. Then f_{M_1} is equivalent either to f_{M_2} or to $-f_{M_2}$, and moreover,

$$A' f_{M_2} A = \det A \cdot f_{M_1}, \tag{18}$$

where A' is the transpose of A (we identify a form with the symmetric matrix which defines it).

Proof. Since M_1 and M_2 are conjugate, they have one and the same characteristic polynomial. By the definition of f_M we have

$$\begin{aligned} f_{M_2}(A\mathbf{v}) &= \det(M_2A\mathbf{v}, (M_2^2 - k_1M_2)A\mathbf{v}, \ldots, A\mathbf{v}) \\ &= \det(AM_1\mathbf{v}, A(M_1^2 - k_1M_1)\mathbf{v}, \ldots, A\mathbf{v}) \\ &= \det A \cdot f_{M_1}(\mathbf{v}), \end{aligned}$$

which is equivalent to (18). ■

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